On the Rayleigh-Bénard problem in the continuum limit: Effects of temperature differences and model of interaction

Avshalom Manela and Itzchak Frankel
Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel

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The transition to convection in the Rayleigh-Bénard problem at small Knudsen numbers is studied via a linear temporal stability analysis of the compressible "slip-flow" problem. Considering a power-law (variable hard-sphere) model of interaction our analysis indicates that for sufficiently large Froude numbers "softer" potentials result in less unstable systems. At small Froude numbers this trend is reversed, i.e., the "softer" interaction potentials correspond to a more unstable response. These results are discussed in terms of the opposing mechanisms of thermal expansion and compressibility. We carry out an asymptotic expansion for small temperature differences and establish the principle of exchange of stabilities for this limit. A singularity appears in this limit when compressibility effects are dominant. © *2005 American Institute of Physics*.
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The onset of Rayleigh-Bénard (RB) convection is a classical problem in hydrodynamic stability theory and has been studied extensively within the framework of the Boussinesq approximation,[1] wherein both relative temperature differences and density variations owing to compressibility effects are assumed small. These assumptions may not apply when considering the RB convection in rarefied gases. The problem has been studied in recent years mainly by means of the direct simulation Monte Carlo (DSMC) method.[2–4] The numerical simulations follow the evolution of the system through its terminal state which, in turn, serves to classify the system response as stable or unstable. The above-mentioned studies demonstrate that convection only occurs at small $O(10^{-2})$ Knudsen numbers. However, the artificial "noise" inherent in these simulations makes it difficult to clearly identify and characterize the final states, particularly for parameter combinations in the vicinity of the transition to convection. Furthermore, these simulations may become extremely time consuming in the continuum limit, which obstructs an accurate delineation of the domain of instability. Consequently, explicit results in the literature have been presented only for a hard-sphere gas in a limited number of parameter combinations. In particular, no results have been presented thus far for Knudsen numbers smaller than 10^{-3}.

Recently, Manela and Frankel[5] studied the RB problem in the continuum limit via a linear temporal stability analysis of the compressible "slip-flow" model.[6] Their predictions of the onset of convection for a hard-sphere gas at a specific temperature difference show remarkable agreement with existing DSMC and continuum nonlinear simulations.[4] This agreement suggests the linear analysis as a useful alternative for studying the RB problem, particularly at arbitrarily small Knudsen numbers. The present contribution is thus aimed at examining the effects of varying both the model of molecular interaction and the temperature difference. In the limit of small temperature differences we obtain *inter alia* the familiar Boussinesq-type results of Spiegel.[7,8] Our analysis indicates, however, that this approximation becomes nonuniform when compressibility effects are dominant.

We consider a layer of perfect monatomic gas confined between infinite horizontal walls and heated from below. To render the problem dimensionless the position vector is scaled by D, the distance between the walls, the gas density by a reference value $\bar{\rho}$ [see (6)], and the temperature by T_h, the absolute temperature of the lower (hot) wall. Fluid velocity is scaled by the mean thermal speed $U_{\text{th}}=(2RT_h)^{1/2}$ (wherein R is the gas constant) and the pressure is normalized by $\bar{\rho}RT_h$. Shear viscosity and heat conductivity are normalized by their respective values, μ_h and κ_h, at T_h. The problem is governed by the continuity, Navier-Stokes, and energy equations together with the perfect gas equation of state. From the symmetry properties of the subsequent perturbation problem[5] we may, without loss of generality, use a two-dimensional description in the Cartesian coordinates (x_1,x_2) whose origin lies on the lower wall. The x_2 axis is directed vertically upwards and x_1 is a horizontal coordinate. In these coordinates

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0, \tag{1}$$

$$\rho\frac{Du_i}{Dt}=-\frac{1}{2}\frac{\partial p}{\partial x_i}+\overline{\text{Kn}}\frac{\partial}{\partial x_j}\left[2\mu\left(e_{ij}-\frac{1}{3}\frac{\partial u_j}{\partial x_i}\right)\right]-\frac{\rho}{\text{Fr}}\delta_{i2}, \tag{2}$$

$$\rho\frac{DT}{Dt}=\frac{\gamma}{Pr}\overline{\text{Kn}}\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right)-(\gamma-1)p\frac{\partial u_i}{\partial x_i}+2(\gamma-1)\overline{\text{Kn}}\Phi, \tag{3}$$

and

$$p=\rho T. \tag{4}$$

In the above summation of repeated indices is implied, D/Dt denotes the material derivative and δ_{ij} is the Kronecker delta. Also appearing in (2) and (3) are e_{ij}, the rate-of-strain tensor, and Φ, the rate of dissipation. The dimensionless parameters appearing in the equations are $\overline{\mathrm{Kn}}=\mu_h/\bar{\rho}U_{th}D$, related to the Knudsen number [see (8) *et seq.*]; the Froude number, $\mathrm{Fr}=U_{\mathrm{th}}^2/gD$, describing the relative magnitudes of gas inertia and gravity, respectively; the Prandtl number, $\mathrm{Pr}=\mu_h c_p/\kappa_h$, and $\gamma=c_p/c_v$, the ratio of specific heat at constant pressure and volume, respectively. For a perfect monatomic gas $\gamma=5/3$ and $\mathrm{Pr}=2/3$.

We assume an inverse power-law central-force interaction between the gas molecules, $F\propto r^{-(s+1)}$, wherein r is the separation between the centers of the molecules and $4\leq s<\infty$ (the limits corresponding to Maxwell molecules and hard spheres, respectively). For this interaction law, the Chapman-Enskog scheme[9] yields the dimensionless transport coefficients[10]

$$\mu(T)=\kappa(T)=T^{1/2+2/s}. \tag{5}$$

The above equations are supplemented by the normalization condition

$$\int \rho\, dx_1 dx_2 = 1, \tag{6}$$

specifying the total amount of gas between the walls, and by the boundary conditions

$$u_2=0, \quad u_1=\pm\zeta\frac{\partial u_1}{\partial x_2}, \quad T=\begin{bmatrix}1\\ R_T\end{bmatrix}\pm\tau\frac{\partial T}{\partial x_2} \quad \text{at } x_2=\begin{bmatrix}0\\ 1\end{bmatrix}, \tag{7}$$

respectively, imposing the vanishing of the normal velocity component and specifying the magnitudes of velocity slip and temperature jump at the lower ($x_2=0$) and upper ($x_2=1$) walls. In (7), $R_T=T_c/T_h$ denotes the ratio of cold- and hot-wall temperatures, $\zeta=1.1466\mathrm{Kn}$ and $\tau=2.1904\mathrm{Kn}$, wherein Kn is the Knudsen number representing the ratio of l, the mean free path, and the macroscopic scale D. We here make use of the slip (ζ) and jump (τ) coefficients calculated[11] using the Bhatnagar-Gross-Krook (BGK) model and neglect the slight (5% at most) dependence of these coefficients upon the model of interaction.[12–14]

To relate the parameters Kn and $\overline{\mathrm{Kn}}$ appearing in the above, the mean free path needs to be expressed in terms of hydrodynamic properties of the gas. For molecules other than hard spheres, there is a certain degree of arbitrariness in the selection of the appropriate (convergent) effective cross section. For the prevailing variable hard-sphere (VHS) model[16] (based on the viscosity cross section) one obtains

$$\overline{\mathrm{Kn}}=G\,\mathrm{Kn}, \quad G=\frac{5\sqrt{\pi}}{16}\frac{3s^2}{(3s-2)(s-1)}\bar{T}^{-2/s}. \tag{8}$$

In (8) $\bar{T}$ denotes the mean gas temperature [see (10)]. It is worthwhile to note that the selection of other models appearing in the literature[15] will only affect the resulting explicit functional form of G.

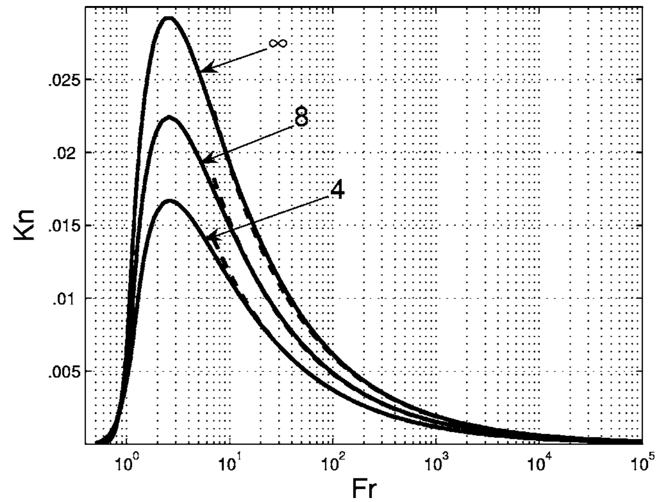

FIG. 1. Neutral curves in the (Fr,Kn) plane for $R_T=0.1$ at the indicated values of s (solid lines). The dashed asymptotes correspond to constant values of Ra_m, the modified Rayleigh number [see (12)].

The above problem (1)–(7) possesses the steady "pure-conduction" (i.e., $u_i^{(0)}=0$) solution

$$T^{(0)}=(Ax_2+B)^{2s/(3s+4)},$$
$$\rho^{(0)}=\frac{C}{T^{(0)}}\exp\left[-\frac{2(3s+4)}{A\mathrm{Fr}(s+4)}T^{(0)^{(s+4)/2s}}\right], \tag{9}$$

in which the constants A, B, and C are determined by use of (6) and (7). $\bar{T}$ in (8) is then obtained as

$$\bar{T}=\int_0^1 T^{(0)}dx_2. \tag{10}$$

The linear temporal stability of this pure-conduction reference state is analyzed assuming that it is perturbed by small spatially harmonic perturbations. Accordingly, each of the above-mentioned fields is generically represented by the sum

$$F=F^{(0)}(x_2)+\phi^{(1)}(x_2)\exp[ikx_1+\omega t], \tag{11}$$

wherein $F^{(0)}$ denotes the steady reference state and k is the real wave number. Substituting (11) into (1)–(7) and neglecting terms nonlinear in the perturbations we obtain a linear problem. With minor modifications the resulting problem is the same as (11)–(15) of Ref. 5 and is therefore not explicitly presented here. The dispersion relation $\omega=\omega(k;\mathrm{Kn},\mathrm{Fr},R_T)$ is obtained by means of the Chebyshev collocation method.[17] Throughout the entire domain of parameters, our calculations invariably yield real valued growth rates. Thus convection appears via stationary perturbations, $\omega=0$. We will later on establish the principle of exchange of stabilities in the limit $R_T\rightarrow 1$.

Figure 1 presents the effects of the parameter s on the convection domain in the (Fr,Kn) plane for $R_T=0.1$. The solid lines are the neutral curves for the limiting cases $s=4$ and ∞, as well as the more realistic[15] intermediate case $s=8$. Following Ref. 3 we define the modified Rayleigh number

$$\mathrm{Ra}_m = \frac{8}{3G^2(1+R_T)^2}\frac{1-R_T}{\mathrm{Fr}\,\mathrm{Kn}^2} \tag{12}$$

based on the arithmetic mean of wall temperatures. At $\mathrm{Fr} \gg 1$, pressure is nearly uniform across the gas [see (4) and (9)]. Compressibility effects are therefore negligible and the various neutral curves become asymptotic to the dashed lines corresponding to constant values of Ra_m ($\approx 1830, 1800, 1773$ for $s=4,8,\infty$, respectively). With diminishing Fr and growing importance of compressibility effects, the neutral curves deviate to increasingly larger values of Ra_m reaching a cutoff at a finite Fr (see Fig. 3 *et seq.*) where compressibility effects become dominant.

With the exception of the neighborhood of the smallest Fr values ($\lesssim 0.85$), i.e., as long as compressibility effects are not dominant, the "softening" of molecular interactions (i.e., diminishing s) has a stabilizing influence which is manifested in the increasing critical values of Ra_m and the diminishing of the maximal values of Kn allowed in the convection domain. According to Frölich *et al.*,[19] the enhanced RB stability relative to the Boussinesq approximation is associated with the variation of κ and μ with T across the fluid. From (5) we see that, for a given $R_T < 1$, the variability of the transport coefficients is indeed increasing with diminishing s.

In contrast to the above, diminishing s is seen to have a destabilizing effect at the left portion of the neutral curves. In this domain of small Fr ($\lesssim 0.85$), compressibility effects dominate thermal expansion. Transition to convection may only occur provided that the adiabatic expansion of a fluid element rising through the reference hydrostatic pressure field reduces its density below the ambient reference value.[18] In the present dimensionless notation, this yields the condition

$$\Delta(x_2) = \frac{dT^{(0)}}{dx_2} + \frac{4}{5\mathrm{Fr}} < 0. \tag{13}$$

Owing to the functional form of the dependence (5) of κ upon T and s, for given Kn and R_T we find that $dT^{(0)}/dx_2$ is diminishing with s. Hence (13) may be satisfied at smaller Fr, which extends the lower left portion of the convection domain.

To study the effects of the wall-temperature ratio we begin with an approximation for a small temperature difference. Introduce the small parameter

$$\epsilon = 1 - R_T \ll 1. \tag{14}$$

The dominant balance of the respective effects of fluid viscosity and heat conductivity, external field, and unsteadiness in the perturbation problem, yields the distinguished limit process

$$\mathrm{Kn} = a\epsilon, \quad \mathrm{Fr} = b/\epsilon, \quad \text{and} \quad \omega = \bar{a}\sigma\epsilon, \tag{15}$$

wherein a, b, and σ are fixed when $\epsilon \to 0$ and $\bar{a} = Ga$ [see (8)].[20] To $O(\epsilon)$ the reference temperature and density distributions (9) are linear functions of x_2. The perturbation problem satisfied by $u_2^{(1)}$, the leading-order vertical velocity perturbation, consists of the equation

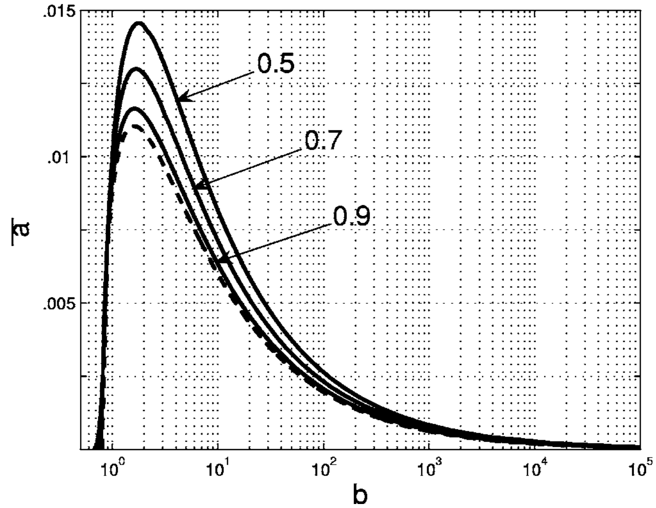

FIG. 2. Effects of R_T, the temperature ratio, on the neutral curves in the $(b,\bar{a})$ plane for a hard-sphere gas. The dashed line corresponds to $\overline{\mathrm{Ra}} \approx 1708$.

$$\left(\frac{d^2}{dx_2^2} - k^2\right)\left(\frac{d^2}{dx_2^2} - k^2 - \sigma\right)\left(\frac{d^2}{dx_2^2} - k^2 - \frac{2}{3}\sigma\right)u_2^{(1)} = -\overline{\mathrm{Ra}}k^2 u_2^{(1)}, \tag{16}$$

together with the boundary conditions

$$u_2^{(1)} = 0,$$
$$\frac{du_2^{(1)}}{dx_2} = 0, \quad \left(\frac{d^2}{dx_2^2} - k^2\right)\left(\frac{d^2}{dx_2^2} - k^2 - \sigma\right)u_2^{(1)} = 0 \quad \text{at } x_2 = 0,1. \tag{17}$$

The resulting problem is governed by the single parameter

$$\overline{\mathrm{Ra}} = \frac{2\beta}{15\bar{a}^2 b^2}, \quad \beta = 5b - 4. \tag{18}$$

The $O(\epsilon)$ dimensionless excess of the absolute-temperature gradient over the adiabatic gradient, $4/5\mathrm{Fr}$, is thus proportional to β. When $\overline{\mathrm{Ra}}$ is replaced by the classical Rayleigh number, (16) and (17) become identical to the perturbation problem obtained within the framework of the Boussinesq approximation.[1] Immediate corollaries of this equivalence are the real values of ω (i.e., the "principle of exchange of stabilities") and the critical values $\overline{\mathrm{Ra}} \approx 1708$ and $k \approx 3.117$ for the onset of convection.

Figure 2 presents the effects of the temperature ratio on the transition to convection for a hard-sphere gas ($s \to \infty$). The solid lines correspond to numerically obtained neutral curves at the indicated values of R_T ($=1-\epsilon$). The dashed line corresponds to $\overline{\mathrm{Ra}} \approx 1708$ which in the plane of $(b,\bar{a})$, the respective generalized Fr and Kn [see (15)] represents the universal (for all s) asymptote to the neutral curve as $R_T \to 1$. It is worthwhile to note that as $R_T \to 1$, the convection domain is confined to vanishingly small Knudsen numbers, which render DSMC simulations rather impractical for $R_T \gtrsim 0.9$.

Results similar to the above (16) and (17) have previously been obtained by Spiegel and co-workers[7,8] in the context of the stability of a thin layer of a perfect gas confined

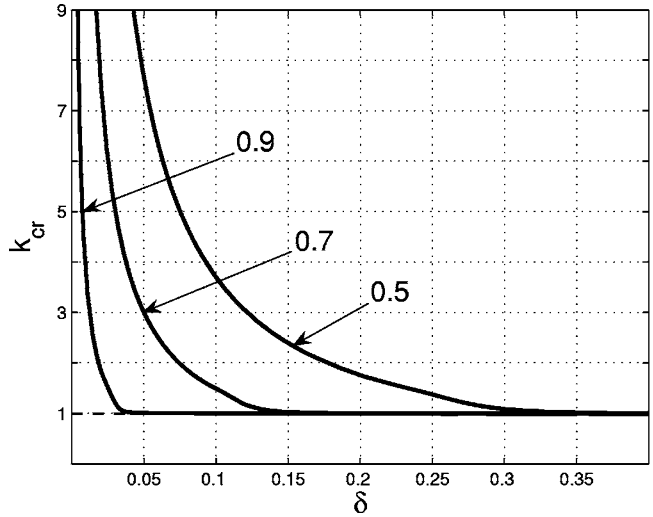

FIG. 3. Variation of k_{cr}, the dimensionless critical wave number (normalized by the Boussinesq value), in the vicinity of Fr_0, the cutoff Froude number, for a hard-sphere gas at the indicated values of R_T.

between free planar boundaries as a model of a polytropic compressible atmosphere. Constant transport coefficients and steady convection have been assumed at the outset. Their results are presented in terms of $\overline{\mathrm{Ra}}$, the normalized width of the layer (which is the counterpart of our $1-R_T$), and the polytropic index m, which is related to the present β via

$$\beta=\frac{6-4m}{5\mathrm{Fr}(m+1)}. \qquad (19)$$

When considering the onset of convection in the respective limits $R_T \to 1$, or of a thin layer, the necessary condition (13) yields $\beta > 0$ which, in turn, requires that $m < 3/2$. This restriction has apparently been left unnoticed in Refs. 7 and 8, where $\overline{\mathrm{Ra}} \approx 1708$ is regarded as the critical value for the onset of convection in a thin layer irrespective of m (see Fig. 1 and Table 1 in Ref. 8).

Inspection of Fig. 2 reveals that convergence to the asymptote is nonuniform in b (the generalized Froude number). The largest relative differences between the respective numerical and asymptotic results take place at the lower-left end of the neutral curves which, as mentioned above, are dominated by compressibility effects. To gain further insight into the behavior of this domain, Fig. 3 presents, at the indicated values of R_T, the variation of k_{cr}, the critical wave number normalized by the value corresponding to the Boussinesq approximation, with $\delta = \mathrm{Fr}/\mathrm{Fr}_0 - 1$, the relative deviation of Fr from the cutoff value Fr_0. [The latter is the smallest value of the Froude number for which, at a given combination of (s, Kn, R_T), the necessary condition (13) is satisfied.] The singular nature of the limit $R_T \to 1$ is reflected in that, for all $1-R_T > 0$, k_{cr} diverges at some (however small) neighborhood of $\delta > 0$. Calculations of the perturbation velocity field (see Figs. 2 and 3 in Ref. 4 and Fig. 3 in Ref. 5) show that, with diminishing δ, the resulting convection becomes confined to an increasingly narrower layer of the fluid adjacent to the upper (cold) wall. Analysis of this singular limit of dominant compressibility effects is currently under way.

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