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From the Generalized Boussinesq Approximation to the Marginally Super-Adiabatic Limit

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The prevailing view of the Rayleigh-Bénard problem in compressible fluids is that for small temperature differences the Boussinesq approximation holds, provided that it is based on a modified Rayleigh number incorporating the potential-temperature gradient. However, for small values of the latter, the onset of convection is characterized by distinct non-Boussinesq features. We consider the linear temporal stability problem and identify the origin of the nonuniformity in the convection and compression-work terms of the perturbation energy balance. We thereby regularize the transition with diminishing potential-temperature gradient from the generalized Boussinesq approximation to the limit when the temperature gradient is only marginally super-adiabatic. It is demonstrated that this transition is accomplished in two phases. Initially, the critical Rayleigh number rapidly increases, which is accompanied by only slight variations of the corresponding wave number. Subsequently, with further diminishing potential-temperature gradients, the critical wave number rapidly increases as well, and the resulting convection becomes effectively confined to a narrowing fluid layer adjacent to the upper wall.

Keywords Compressibility; Generalized Boussinesq approximation; Rayleigh-Bénard instability; Super-adiabatic limit

Introduction

The Rayleigh-Bénard (RB) problem has been studied extensively within the framework of the Boussinesq approximation (Chandrasekhar, 1961). This approximation models an essentially incompressible fluid possessing constant viscosity and heat conductivity. Only density variations resulting from thermal expansion are considered and then only in the buoyancy term in the equation of motion. The resulting stability problem is governed by a single parameter, the Rayleigh number (Ra), representing the relative effects of buoyancy on the one hand and fluid viscosity and heat conductivity on the other hand. For a fluid confined between infinite planar walls, the Boussinesq approximation predicts the onset of convection beyond the critical value $Ra_{cr} \approx 1708$ and characterized by a dimensionless wave number (scaled by D, the gap width) $k_{cr} \approx 3.117$.

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When allowing for compressibility effects, it is necessary for the onset of convection that the adiabatic expansion of a fluid element rising through the reference hydrostatic pressure field reduces its density below the ambient value (Landau and Lifshitz, 1959). This is usually expressed as a condition on the (negative) vertical reference temperature gradient requiring that it be smaller than the adiabatic gradient corresponding to the ambient hydrostatic pressure distribution. This requirement becomes the dominant condition for the onset of convection in natural situations (e.g., on atmospheric scales) where the Rayleigh number greatly exceeds the critical value (Tritton, 1988) as well as in laboratory experiments with near-critical fluids (owing to their large compressibility; Kogan and Meyer, 2001). For a perfect monatomic gas we thus require that

$$Fr > Fr_0 = -\frac{4}{5}\left(\frac{dT^{(0)}}{dy}\right)^{-1} \tag{1}$$

wherein $T^{(0)}(y)$ denotes the dimensionless temperature distribution normalized by T_h, the temperature of the hot wall, and the vertical coordinate y is scaled by D. On the left-hand side of Equation (1) appears the Froude number, $Fr = 2RT_h/gD$ (wherein R is the universal gas constant and g is the gravitational acceleration), representing the relative magnitudes of thermal and gravitational effects.

Assuming a small temperature difference and uniform viscosity and heat conductivity across the fluid, the RB problem for a compressible fluid has been studied by Jeffreys (1930), Spiegel (1965), and Giterman and Shteinberg (1970), among others. Based on these analyses the prevailing view is that under these conditions the Boussinesq approximation applies to a compressible fluid provided that a generalized Rayleigh number based on the potential (i.e., the excess of the actual over the adiabatic) temperature gradient is employed (Tritton, 1988). The RB problem for a compressible fluid subject to an arbitrary temperature difference has been addressed in the continuum limit of slightly rarefied gases (Sone et al., 1997; Stefanov et al., 2002; Manela and Frankel, 2005a). The stability problem is here characterized by a pair of parameters, e.g., Fr and the Knudsen number, rather than just by Ra. A unique feature of this problem is that with diminishing Fr a distinctly non-Boussinesq behavior emerges. Convection is effectively confined to a narrow layer of fluid adjacent to the upper (cold) wall and is characterized by wave numbers significantly larger than the above-mentioned $k_{cr}(\approx 3.117)$ (Stefanov et al., 2002; Manela and Frankel, 2005a,b). Furthermore, this trend persists even at small temperature differences (Manela and Frankel, 2005b), which is at variance with the above view regarding validity of the generalized Boussinesq approximation. Manela and Frankel (2006) have related this nonuniformity to the fact that with decreasing small Fr, Equation (1) is satisfied only over a diminishing portion of the fluid domain. They have thus substantiated the non-Boussinesq features of the limit $Fr \to Fr_0$ when the reference-temperature gradient is only marginally super-adiabatic (i.e., when, Equation (1) is only locally satisfied at the upper (cold) wall).

The goal of the present contribution is to regularize and rationalize the transition between the above limits. In the next section we formulate the linearized perturbation problem. The breakdown of the generalized Boussinesq approximation and the transition to the limit $Fr \to Fr_0$ are analysed, the results are illustrated and discussed, and some concluding comments are made.

Formulation of the Problem

We consider a perfect monatomic gas confined between infinite horizontal walls and heated from below. Gas density, fluid velocity, and the pressure are scaled by $\bar{\rho}$, the average density, $U_{th} = (2RT_h)^{1/2}$, and $\bar{\rho}RT_h$, respectively. Shear viscosity and heat conductivity are normalized by μ_h and κ_h, their respective values at T_h. The dimensionless problem is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2}$$

together with the Navier-Stokes

$$\rho \frac{D\mathbf{u}}{Dt} = -\frac{1}{2}\nabla p + \frac{1}{Re}\nabla \cdot \left[2\mu\left(e - \frac{1}{3}\nabla \mathbf{u}\right)\right] - \frac{\rho}{Fr}\hat{\mathbf{j}} \tag{3}$$

and energy

$$\rho \frac{DT}{Dt} = \frac{5}{2Re}\nabla \cdot (\kappa \nabla T) - \frac{2}{3}p\nabla \cdot \mathbf{u} + \frac{4}{3Re}\Phi \tag{4}$$

equations as well as the perfect gas equation of state

$$p = \rho T \tag{5}$$

In the above equations D/Dt denotes the material derivative, $Re = \bar{\rho}U_{th}D/\mu_h$ is the Reynolds number, $\hat{\mathbf{j}}$ is the unit vector pointing vertically upwards, $\mathbf{u}$ is the velocity vector, and e and Φ denote the rate-of-strain tensor and the rate of dissipation, respectively. For the dimensionless transport coefficients we assume the power law $\mu(T) = \kappa(T) = T^{1/2}$ (corresponding to the hard-sphere model in gas-kinetic theory; Chapman and Cowling, 1970). The above equations are supplemented by the normalization condition for ρ and by the boundary conditions imposing no-slip and temperature continuity at the walls.

The above problem possesses the steady “pure conduction” (i.e., $\mathbf{u}^{(0)} = 0$) solution (Stefanov et al., 2002):

$$T^{(0)} = [1 + (R_T^{3/2} - 1)y]^{2/3}, \quad \rho^{(0)} = \frac{C}{T^{(0)}}\exp\left[-\frac{6}{(R_T^{3/2} - 1)Fr}T^{(0)^{1/2}}\right] \tag{6}$$

in which $R_T = T_c/T_h$ is the ratio of the cold- and hot-wall temperatures and C is calculated by use of the above-mentioned normalization condition. The linear temporal stability of this reference state is analyzed assuming that it is perturbed by small spatially harmonic perturbations. By the transverse symmetry of the problem (Manela and Frankel, 2005a) we use a two-dimensional description in the Cartesian coordinates (x, y) whose origin lies on the lower wall and where x is a horizontal coordinate in the wave-vector direction. Accordingly, each of the above-mentioned fields is generically represented by the sum

$$F = F^{(0)}(y) + \phi^{(1)}(y)\exp[ikx + \omega t] \tag{7}$$

wherein $F^{(0)}$ denotes the steady reference state, k is the real wave number, and ω is the (complex-valued) growth rate. Substituting Equation (7) into Equations (2)–(4) and the attendant boundary conditions while neglecting nonlinear terms in the perturbations we obtain a linear perturbation problem. For brevity we explicitly present here only the continuity equation

$$\omega\rho^{(1)} + \frac{d\rho^{(0)}}{dy}v^{(1)} + \rho^{(0)}\left[\frac{dv^{(1)}}{dy} + f^{(1)}\right] = 0 \tag{8}$$

and the perturbation energy equation

$$\begin{aligned}\omega\rho^{(0)}T^{(1)} + \rho^{(0)}v^{(1)}\frac{dT^{(0)}}{dy} &= -\frac{2}{3}\rho^{(0)}T^{(0)}\left[\frac{dv^{(1)}}{dy} + f^{(1)}\right] + \\ \frac{5}{2Re}\left\{\frac{d\kappa^{(0)}}{dy}\frac{dT^{(1)}}{dy} + \frac{dT^{(0)}}{dy}\frac{d\kappa^{(1)}}{dy}\right. &+ \left.\kappa^{(0)}\left[\frac{d^2T^{(1)}}{dy^2} - k^2T^{(1)}\right] + \kappa^{(1)}\frac{d^2T^{(0)}}{dy^2}\right\}\end{aligned} \tag{9}$$

wherein $\mathbf{u}^{(1)} = (u^{(1)}, v^{(1)})$ is the perturbation velocity vector, $f^{(1)} = iku^{(1)}$, $\kappa^{(0)} = T^{(0)^{1/2}}$, and $\kappa^{(1)} = (1/2)T^{(0)^{-1/2}}T^{(1)}$. (The full system of equations can be found, for example, in (11)–(14) of Manela and Frankel, 2005a). These are supplemented by the perturbation equations of motion (see Equations (14) and (15)) and the boundary conditions prescribing the vanishing of temperature and velocity perturbations at $y = 0$, 1. In agreement with our numerical calculations (see at the beginning of the Results section), we hereafter focus on a stationary (i.e., $\omega = 0$) transition to convection.

Analysis

Breakdown of the Generalized Boussinesq Limit

Assuming an asymptotically small temperature difference

$$\epsilon = 1 - R_T \ll 1 \tag{10}$$

we consider the limit process

$$Fr = \epsilon^{-1}Fr^* \quad \text{and} \quad Re = \epsilon^{-1}Re^* \tag{11}$$

wherein Fr^* and Re^* are fixed when $\epsilon \to 0$. Expanding the pure-conduction reference state (6) into power series in ϵ we obtain

$$F^{(0)}(y) \sim 1 + \epsilon F_1^{(0)}(y) + \epsilon^2 F_2^{(0)}(y) \tag{12}$$

in which $F_1^{(0)}$ and $F_2^{(0)}$ are linear and quadratic functions of y, respectively. Substituting Equations (10)–(12) in the above perturbation problem while assuming that all perturbations appearing in Equation (7) are $\phi^{(1)}(y) \sim O(1)$ we obtain from the $O(1)$ continuity equation

$$\frac{dv^{(1)}}{dy} + f^{(1)} = 0 \tag{13}$$

and from the same order of the x-momentum equation

$$T^{(1)} + \rho^{(1)} = 0 \tag{14}$$

Combining the x- and y-momentum equations to eliminate the pressure terms we obtain at the $O(\epsilon)$ leading order

$$\frac{1}{Re^*}\left[\frac{d^3 f^{(1)}}{dy^3} + k^2\left(\frac{d^2 v^{(1)}}{dy^2} - \frac{df^{(1)}}{dy} - k^2 v^{(1)}\right)\right] - \frac{k^2}{Fr^*}\rho^{(1)} = 0 \tag{15}$$

and from the $O(\epsilon)$ energy equation

$$\left(\frac{2}{3}\frac{d\rho_1^{(0)}}{dy} - \frac{dT_1^{(0)}}{dy}\right)v^{(1)} + \frac{5}{2Re^*}\left(\frac{d^2 T^{(1)}}{dy^2} - k^2 T^{(1)}\right) = 0 \tag{16}$$

From Equations (13)–(16) together with the above-mentioned boundary conditions we obtain for $v^{(1)}$, the vertical component of the perturbation velocity, the Boussinesq-type problem consisting of the equation

$$\left(\frac{d^2}{dy^2} - k^2\right)^3 v^{(1)} = -Ra_1 k^2 v^{(1)} \tag{17}$$

which is supplemented by

$$v^{(1)} = 0, \quad \frac{dv^{(1)}}{dy} = 0, \quad \left(\frac{d^2}{dy^2} - k^2\right)^2 v^{(1)} = 0 \quad \text{at } y = 0, 1 \tag{18}$$

The single parameter governing the problem is the generalized Rayleigh number

$$Ra_1 = \frac{2}{15}\frac{Re^{*2} G_1^{(0)}}{Fr^*} \tag{19}$$

wherein

$$G_1^{(0)} = \frac{2}{3}\frac{d\rho_1^{(0)}}{dy} - \frac{dT_1^{(0)}}{dy} = \frac{5Fr^* - 4}{Fr^*} \tag{20}$$

is proportional to the $O(\epsilon)$ leading potential-temperature gradient (Spiegel, 1965; Manela and Frankel, 2005b).

Validity of the above limit depends upon the nonvanishing of $G_1^{(0)}$. However, inspecting Equation (1) in conjunction with Equation (12) we obtain

$$Fr_0 \sim \frac{4}{5\epsilon}[1 + O(\epsilon)] \tag{21}$$

Thus, with diminishing $Fr \to Fr_0$, $Fr^* \to 4/5$, and the generalized Boussinesq approximation becomes non uniform (Manela and Frankel, 2005b). This nonuniformity may be traced back to the energy Equation (16), where $G_1^{(0)}$ is multiplying $v^{(1)}$.

Evidently, with the vanishing of $v^{(1)}$ from the energy equation, the above perturbation scheme breaks down, admitting only trivial solutions.

Regularization of the Transition between Limits

To examine the transition from the generalized Boussinesq limit to the marginally super-adiabatic limit with Fr approaching Fr_0 we define the small parameter

$$\delta = \frac{Fr}{Fr_0} - 1 \tag{22}$$

Subsequent analysis of the transition to the super-adiabatic limit is governed by the relative magnitude of δ compared to ϵ.

The vertical perturbation velocity $v^{(1)}$ is introduced into the energy Equation (9) through both the convection term on the left-hand side the compression-work (i.e., the first) term on the right-hand side. When the latter contribution is expressed by means of the equation of continuity (8) we find that $v^{(1)}$ appears in the energy equation multiplied by

$$G^{(0)} = \frac{2}{3}\frac{d\rho^{(0)}}{dy} - \frac{dT^{(0)}}{dy} \tag{23}$$

(Thus, see Equation (12), $G_1^{(0)}$ (20) is the coefficient of the leading $O(\epsilon)$ approximation of $G^{(0)}$ in the limit $\epsilon \to 0$ when $\delta \sim O(1)$ is fixed.) In terms of the above-defined δ and making use of the reference state (12) for $\epsilon \to 0$ we obtain

$$G^{(0)} \sim \frac{5}{3}\frac{\epsilon\delta}{1+\delta} + \epsilon^2\left[y\left(\frac{1}{6}\Delta^2 - \frac{7}{6}\Delta + \frac{5}{2}\right) - \frac{1}{12}(\Delta + 5)\right] \tag{24}$$

wherein $\Delta = 5(1+\delta)^{-1}$. In the following we examine the variation with δ of the onset of convection in the limit $\epsilon \to 0$.

To begin with we allow δ to diminish within the range $O(\epsilon) < \sim \delta < \sim O(1)$. To obtain a consistent dominant balance we modify the above limit process replacing Equation (11) by Equation (22) and

$$Re = \frac{a_1}{\epsilon\delta^{1/2}} \tag{25}$$

in which a_1 is fixed when $\epsilon \to 0$. Furthermore, unlike the generalized Boussinesq limit, we need to distinguish between the respective leading orders of the various perturbations. Normalizing $v^{(1)}$ to be $O(1)$, $f^{(1)}$ remains $O(1)$ as well, whereas for the temperature and density perturbations we assume $T^{(1)}$, $\rho^{(1)} \sim O(\delta^{1/2})$. The resulting continuity, x-momentum, and combined-momentum perturbation equations remain the same as the above Equations (13)–(14) and (15), respectively (though the orders of the latter pair are now $O(\delta^{1/2})$ and $O(\epsilon\delta^{1/2})$, respectively). In the $O(\delta\epsilon)$ leading energy balance $G_1^{(0)}$ (20) needs to be replaced by $G^{(0)}$ (24) as the coefficient of $v^{(1)}$. From these we obtain for $v^{(1)}$ the equation

$$\left(\frac{d^2}{dy^2} - k^2\right)^3 v^{(1)} = -\frac{a_1^2 G^{(0)^*}}{2(1+\delta)} k^2 v^{(1)} \tag{26}$$

which is supplemented by the boundary conditions (18). In (26) $G^{(0)^*} = G^{(0)}/\epsilon\delta$.

Evidently, for $\delta \sim O(1)$ Equations (25) and (26) reduce to the above generalized Boussinesq approximation. The same is true for $\delta \sim o(1)$ as well, provided that $\epsilon/\delta \sim o(1)$. For $\delta \sim O(\epsilon)$, however, both terms in $G^{(0)}$ (24) become comparable. Consequently, $G^{(0)^*}$ is no longer a constant but rather a function of y:

$$G^{(0)^*} \sim \frac{5}{3} + \frac{5}{6}\frac{\epsilon}{\delta}(y-1) \tag{27}$$

which manifests the departure from the above generalized Boussinesq approximation. Solutions to the eigenvalue problem governed by Equation (26) in conjunction with Equation (27) are discussed below.

With δ further diminishing to $\delta \sim o(\epsilon)$ the limit process embodied in Equation (25) together with the accompanying scaling of the perturbations become nonuniform. This is manifested in the breakdown of the dominant balance in the energy Equation (16). Thus, the conduction terms decrease as $\epsilon\delta$ whereas $G^{(0)}v^{(1)}$ apparently remains $O(\epsilon^2)$. To obtain a consistent balance we again modify the limit process by rescaling Re and k and defining the "inner" variable Y through

$$Re = \frac{\epsilon}{\delta^{5/2}}a_2, \quad k = \frac{\epsilon}{\delta}l, \quad \text{and} \quad Y = k(1-y) \quad (0 \le Y \le k) \tag{28}$$

respectively, where a_2, l, and Y are fixed for $\epsilon \to 0$. (By definition, $f^{(1)}$ is now $O(\epsilon/\delta)$. Otherwise, the leading orders of the various perturbations remain unchanged.) Expanding the reference state (12) about the regular point ($y = 1$, $Fr = Fr_0$) and substituting this expansion into the perturbation problem (see Equation (8) and following) we obtain for the leading-order vertical perturbation velocity in terms of the inner variable the eigenvalue problem

$$\left(\frac{d^2}{dY^2} - 1\right)^3 v^{(1)} = (A_0 Y - B_0)v^{(1)} \tag{29}$$

together with

$$v^{(1)} = 0, \quad \frac{dv^{(1)}}{dY} = 0, \quad \text{and} \quad \left(\frac{d^2}{dY^2} - 1\right)^2 v^{(1)} = 0 \quad \text{at } Y = 0,\ k \tag{30}$$

In Equation (29) we denote $A_0 = (5/12)a_2^2/l^5$ and $B_0 = (5/6)a_2^2/l^4$. For $\delta/\epsilon \sim O(1)$, Equation (28)–(30) are equivalent to Equations (25), (26), and (18), respectively, hence the present limit may actually serve to describe the entire range $\delta/\epsilon < \sim O(1)$. The resulting eigenvalue problem for a_2 is governed by the two parameters l and δ/ϵ.

The general solution of Equation (29) is represented via the superposition (Duty and Reid, 1964)

$$v_0^{(1)}(Y) = \Sigma_{n=0}^{5}\, g_n f_n(Y) \tag{31}$$

where g_n ($n=0, 1,\ldots,5$) are arbitrary (complex) constants and

$$f_n(Y) = \int_{C_n} \exp\left[-\frac{1}{A_0}\left(\frac{1}{7}z^7 - \frac{3}{5}z^5 + z^3 + (B_0 - 1)z\right) + Yz\right] dz \tag{32}$$

are generalized Laplace integrals in the complex z-plane. The contours C_n (Granoff and Bleistein, 1972) originate and terminate at $z \to \infty$ within any of the seven sectors $-(\pi/14)(1-4n) < \arg z < (\pi/14)(1+4n)$, ($n=0, 1,\ldots, 6$).

Substituting the (numerically evaluated) functions $f_n(Y)$ ($n=0, 1,\ldots, 5$) and the respective derivatives at $Y=0$, k into Equation (30), we obtain a system of six homogeneous linear algebraic equations for g_n ($n=0, 1,\ldots, 5$). The requisite dispersion relation $a_2 = a_2(l, \delta/\epsilon)$ is thereby obtained as the lowest branch of the characteristic equation resulting from the vanishing of the system determinant. Making use of the relation between Ra_1, Fr, and Re, the generalized Rayleigh number is here expressed as

$$Ra_1 = \frac{5a_1^2}{6(1+\delta)^2} \tag{33}$$

which (see Equations (22) and (25)) for $\delta \sim O(1)$ reduces to Equation (19). Explicit results will be presented in terms of Ra_1, k, and δ/ϵ.

Results and Discussion

Numerical solution of the eigenvalue problem formulated above by means of the Chebyshev collocation method invariably yields real-valued growth rates throughout the entire domain of parameters, as obtained in our previous numerical

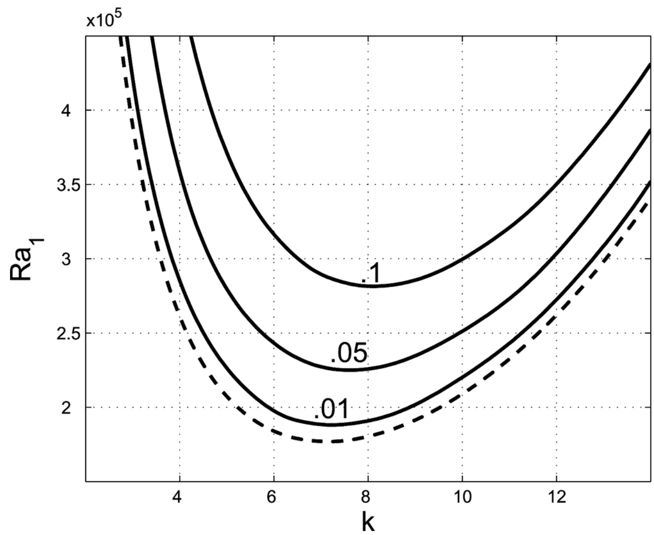

Figure 1. The neutral curves in the (k, Ra_1) plane of wave number and generalized Rayleigh number for $\delta/\epsilon = 1/7$. The solid lines correspond to the numerical solution at the indicated values of ϵ, whereas the dashed curve depicts the $\epsilon \to 0$ asymptote.

investigations (Manela and Frankel, 2005a, b) carried out for a wide range of Froude numbers ($10^{-1} < Fr < 10^6$). This observation supports our focusing in the preceding section on transition to convection through stationary states ($\omega = 0$).

Figure 1 examines convergence of the neutral curve in the limit $\epsilon \to 0$ for $\delta/\epsilon = 1/7$ (fixed). The solid lines result from numerical calculations based on the "exact" problem at the indicated values of ϵ. The dashed line marks the $\epsilon \to 0$ asymptote corresponding to the dispersion relation $Ra_2 = Ra_2(l, \delta/\epsilon)$, where the modified Rayleigh number Ra_2 is obtained from Ra_1 through replacing a_1 in Equation (33) by a_2. (From the definitions (25) and (28) $Ra_2 = Ra_1(\delta/\epsilon)^4$.) It is worthwhile to note that the critical values $Ra_{cr} \approx 1.77 \times 10^5$ and $k_{cr} \approx 7.21$ are much larger than their Boussinesq counterparts.

The transition from the Boussinesq limit to the marginally super-adiabatic limit is described through the variation with diminishing (fixed) δ/ϵ of the neutral curve in the (k, Ra_1) plane. Thus, the solid lines of Figure 2(a) present on a semilogarithmic

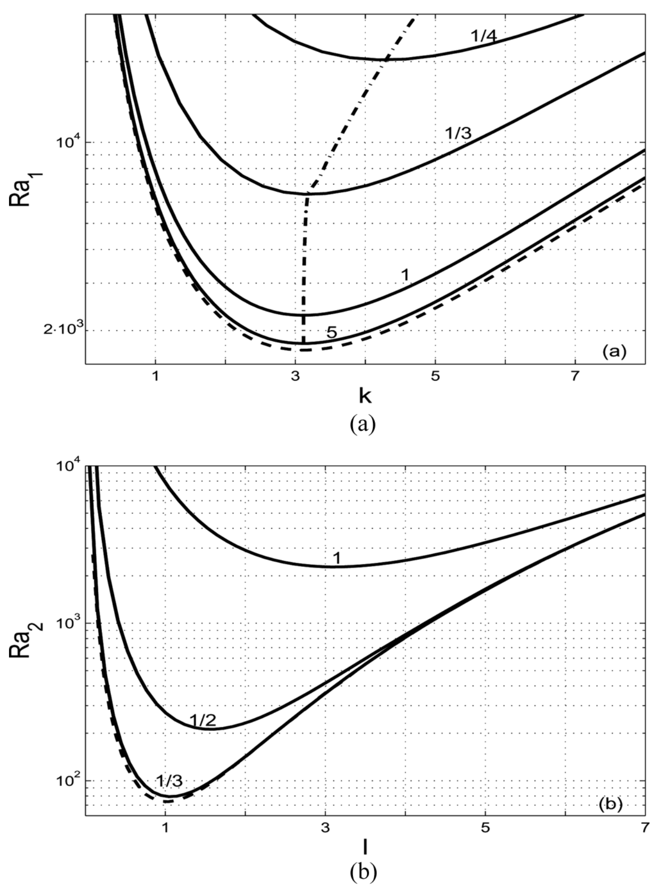

Figure 2. (a) The neutral curves in the (Ra_1, k) plane for $\epsilon \to 0$ and the indicated values of δ/ϵ. The dashed line corresponds to the generalized Boussinesq approximation. The dash-dotted line depicts the variation of the critical conditions $(k_{cr}, Ra_{1_{cr}})$ with δ/ϵ. (b) The neutral curves in the (l, Ra_2) plane for $\epsilon \to 0$ and the indicated values of δ/ϵ. The dashed line marks the $\delta/\epsilon \to 0$ asymptote.

scale the neutral curves $Ra_1 = Ra_1(k)$ corresponding to the dispersion relation $a_2 = a_2(l, \delta/\epsilon)$ obtained in the limit $\epsilon \to 0$ with δ/ϵ fixed at the indicated values. For comparison the generalized Boussinesq approximation (marked by the dashed curve) is presented as well. The latter asymptote is indeed a close approximation for $\delta/\epsilon > \sim 5$. With diminishing δ/ϵ we observe that the critical values of Ra_1 rapidly increase. Down to $\delta/\epsilon \approx 1/3$ this is accompanied by only a slight variation of the corresponding k_{cr}. At this point the dash-dotted critical-conditions curve undergoes a sharp bend. With further decreasing $\delta/\epsilon < 1/3$ k_{cr} rapidly increases, which reflects the emergence of a non-Boussinesq convection pattern (see Figure 3).

To complete the description for $\delta/\epsilon \to 0$, Figure 2(b) presents the neutral curves in the (l, Ra_2) plane. The curve $\delta/\epsilon = 1$ is the same as in Figure 2(a) since for this value $l = k$, $a_2 = a_1$, and hence $Ra_2 = Ra_1$. The dashed asymptote $\delta/\epsilon \to 0$ is nearly indistinguishable from the curve $\delta/\epsilon = 1/3$. This rapid convergence verifies appropriateness of the scaling of the Reynolds number, the wave number, and the inner variable assumed in the limit process (28).

To gain further insight into the above trends for $\delta < \epsilon$, we present in Figure 3 the eigenfunctions $v^{(1)}(y)$ (normalized to unity) of the vertical velocity perturbation at critical conditions $(Ra_{1_{cr}}, k_{cr})$ for $\epsilon \to 0$ and δ/ϵ fixed at the indicated values. As a reference the dashed line marks the generalized Boussinesq eigenfunction, which nearly coincides with the actual $v^{(1)}$ for all $\delta/\epsilon > \sim 1/3$. However, with further diminishing δ convection becomes essentially confined to an ever-narrowing fluid layer adjacent to the upper (cold) wall. The corresponding (critical) wave numbers rapidly grow (as observed in Figure 2(a)). These non-Boussinesq features are most evident in the curve corresponding to $\delta/\epsilon = 1/15$ (the corresponding critical wave number is $k_{cr} \approx 15.3$).

The above trends accompanying diminishing δ/ϵ may be rationalized in terms of the increasing relative importance of the requirement (1) associated with the adiabatic temperature gradient. The critical value of the generalized Rayleigh number thus increases so as to provide the requisite potential temperature gradient. Furthermore,

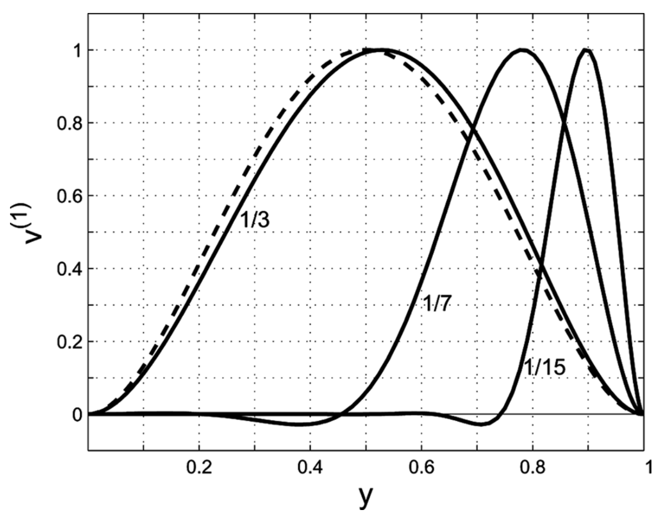

Figure 3. The normalized eigenfunctions of the vertical velocity perturbation for $\epsilon \to 0$ and critical conditions at the indicated values of δ/ϵ. The dashed line marks the generalized Boussinesq limit.

from Equation (1) in conjunction with $T^{(0)}$ in Equations (6) and (12), we find that Equation (1) is satisfied throughout the entire fluid domain only provided that $\delta/\epsilon \geq 1/2$. With δ/ϵ further decreasing, Equation (1) is satisfied within a fluid layer whose width diminishes like δ/ϵ (in agreement with the scaling (28) of the inner variable). Figure 2 indicates that the actual departure from the Boussinesq approximation insofar as the critical wave numbers and eigenfunctions are concerned does not occur until $\delta/\epsilon < \sim 1/3$. This is related to viscous and heat-conduction effects, which are not accounted for by the necessary condition (1). Thus, once convection sets in it extends over a gas layer, which is wider than the expected $\approx \delta/\epsilon$ owing to viscous momentum diffusion to lower fluid layers. At $\delta/\epsilon \approx 1/3$ convection still occupies the entire fluid domain and the Boussinesq eigenfunction and critical wave number are nearly recovered. However, the corresponding $Ra_{cr} \approx 6.2 \times 10^3$ (Figure 2(a)) is already much larger than the Boussinesq counterpart.

Conclusion

We have analyzed a nonuniformity occurring in the generalized Boussinesq approximation under marginally super-adiabatic conditions. We have studied the linearized temporal stability problem and identified the origin of this nonuniformity in the convection and compression-work terms of the perturbation energy balance. It has been demonstrated that the transition between the generalized Boussinesq approximation and the marginally super-adiabatic limit is accomplished in two phases. Initially, the critical Rayleigh number rapidly increases, accompanied by slight variations of the corresponding wave number. Then, with further diminishing potential-temperature gradients, the critical wave number rapidly increases as well, and the resulting convection becomes effectively confined to a narrowing fluid layer adjacent to the upper wall.

Our analysis focuses on the limit where ϵ, the relative temperature difference between the walls, and δ, the super-adiabatic deviation parameter, are asymptotically small. In actual physical situations (e.g., atmospheric phenomena on the scale of hundreds of meters or experiments with near-critical fluids), ϵ is small but finite. Nonetheless, we expect our predictions regarding non-Boussinesq behavior to qualitatively apply at finite (small) δ ($<\epsilon/3$) and corresponding finite (large) Ra_{cr}. This anticipation is supported by the qualitative agreement with results pertaining to finite ϵ (Stefanov et al., 2002; Manela and Frankel, 2005a), showing similar behavior at the onset of instability at small *Fr* in the continuum limit. In particular, Stefanov et al. (2002) have obtained these features (see their Figures 2 and 3) via simulations of the fully nonlinear initial-value problem, which suggests relevance of the present linear analysis to the actual onset of convection at arbitrary ϵ and slightly super-adiabatic conditions.

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